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Fixed Points of Renormalization Group for the Hierarchical Fermionic Model

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A fermionic version of Dyson's hierarchical model is defined. An exact renormalization group transformation is given as a rational transformation of two-dimensional parameter space. Two branches of nontrivial fixed points are described, one of which bifurcates from the trivial "Gaussian" branch. The existence of the thermodynamic limit for these branches of fixed points is investigated.

KEY WORDS: Renormalization group; fixed point; hierarchical model; Grassmann variables.

1. INTRODUCTION

In this paper we discuss a fermionic analog of the hierarchical φ^4-model for which an action of the renormalization group (RG) transformation is described as an exact rational transformation in a two-dimensional parameter space. The simplicity of description of renormalization group flow is explained by a combination of two factors: the local property of hierarchical block-spin transformation and the fact that the spins are elements of Grassmann algebra. Using the fact that local potentials are conserved under block-spin transformation, Bleher and Sinai,[1] Collet and Eckmann,[2] and Gawedzki and Kupiainen[3] carried out rigorous detailed investigations of hierarchical bosonic models. On the other hand using nice combinatoric properties of fermionic models, Gawedzki and Kupiainen[4] and Feldman *et al.*[5] accomplished a rigorous RG analysis of the Gross–Neveu model.

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Our observation is motivated by the work of Dorlas,[6] who proposed a simple fermionic hierarchical model. In distinction to Dorlas' model, the "Gaussian" part of our model is nondegenerate and we study the RG transformation for all possible values of n (size of an elementary block of the hierarchical lattice) and all possible values of RG parameter α. The simplicity of the RG transformation enables us to give an exact description of all branches of fixed points of the renormalization group and to investigate the existence of the thermodynamic limit for these points. One of nontrivial branches bifurcates from the trivial "Gaussian" one and can serve as an elementary illustration of an $\varepsilon = \alpha - 3/2$ (or $\varepsilon = 4 - d$) expansion. It is interesting that the second branch of fixed points does not bifurcate from the "Gaussian" branch and the existence of an analogous branch in the bosonic case remains unknown. The infinite-volume limit exists for some part of this branch, for $n > 13$. The nonexistence of the thermodynamic limit for the nontrivial fixed point in ref. 6 probably is connected with the low value of n in this model.

The global RG flow in the whole plane of coupling constants, the problem of the large-scale limit, and other critical properties of this model will be discussed in a subsequent paper.

2. RG TRANSFORMATION IN HIERARCHICAL $(\bar{\psi}, \psi)^2$ MODEL

We recall some definitions of hierarchical model.

Let $\mathbb{N} = \{1, 2,...\}$, $V_{k,s} = \{j: j \in \mathbb{N}, (k-1)\, n^s < j \leqslant kn^s\}$, $k \in \mathbb{N}$, $s \in \mathbb{N}$, and let $s(i, j) = \min\{s: \text{there is } k \text{ such that } i \in V_{k,s},\ j \in V_{k,s}\}$. The hierarchical distance $d(i, j)$, $i, j \in \mathbb{N}$, is defined by the formula

$$d(i, j) = \begin{cases} 0, & i = j \\ n^{s(i,j)}, & i \neq j \end{cases}$$

Let us consider the 4-component fermionic field $(\bar{\psi}_1(1), \psi_1(i), \bar{\psi}_2(i), \psi_2(i))$, $i \in \mathbb{N}$, where the components are generators of a Grassmann algebra. We shall use the following notations:

$$\bar{\psi}(i) = (\bar{\psi}_1(i), \bar{\psi}_2(i)), \qquad \psi(i) = (\psi_1(i), \psi_2(i))$$
$$\bar{\psi}(i)\, \eta(i) = \bar{\psi}_1(i)\, \eta_1(i) + \bar{\psi}_2(i)\, \eta_2(i), \qquad i \in \mathbb{N}$$

By analogy with the bosonic case the block-spin renormalization group transformation is defined by the formula

$$(\bar{\psi}'(i), \psi'(i)) \equiv r_\alpha(\bar{\psi}, \psi)(i) = n^{-\alpha/2} \sum_{j \in V_{i,1}} (\bar{\psi}(j), \psi(j)) \tag{1}$$

where α is the renormalization group parameter. Then it is easy to see that the "Gaussian" fermionic field with zero mean and binary correlation function

$$\langle \psi_k(i)\,\bar{\psi}_l(j)\rangle = \delta_{k,l} b(i,j), \qquad k, l = 1, 2$$
$$b(i,j) = \frac{1-n^{1-\alpha}}{1-n^{\alpha-2}} d^{\alpha-2}(i,j), \quad i \neq j, \qquad b(i,i) = \frac{1-n^{-1}}{1-n^{\alpha-2}}, \quad \alpha \neq 2 \tag{2}$$

is invariant under the renormalization group transformation. We shall denote this "Gaussian" state by $\langle \cdot \rangle_0$.

Rigorously speaking, $\langle \cdot \rangle_0$ is a state only for $\alpha < 2$, as

$$\langle \psi_{j_1}(i_1)\,\bar{\psi}_{j_1}(i_1)\cdots\psi_{j_m}(i_m)\,\bar{\psi}_{j_m}(i_m)\rangle_0 = \det(\langle \psi_{j_k}(i_l)\,\bar{\psi}_{j_l}(i_l)\rangle_0)_{k,l=1}^m \geqslant 0$$

for any $i_1,\dots, i_m, j_1,\dots, j_m$ if only $\alpha < 2$. For $\alpha > 2$, $\langle \cdot \rangle_0$ is a quasistate, but here we do not attend to this distinction.

Let us redenote $V_{1,N}$ by Λ_N and let $\mathfrak{A}_N$ be the Grassmann subalgebra generated by $4 \cdot n^N$ generators corresponding to this volume.

To study non-"Gaussian" states we use the Gibbs representation of the states. Let us consider, for instance, the restriction of the "Gaussian" state $\langle \cdot \rangle_0$ on the volume Λ_N. Then the following statement is true:

Lemma 1. Let $F(\bar{\psi}, \psi) \in \mathfrak{A}_N$. Then

$$\langle F(\bar{\psi}, \psi\rangle_0 = Z_{0,N}^{-1} \int F(\bar{\psi}, \psi) \exp\{-H_{0,N}(\bar{\psi}, \psi, \alpha)\}\, d\psi\, d\bar{\psi}$$

where the anticommuting integration rule is defined, following Berezin,[7] by setting $\int \psi_i\, d\psi_i = 1$, $\int d\psi_i = 0$

$$H_{0,N}(\bar{\psi}, \psi, \alpha) = \sum_{i,j \in \Lambda_N} d_{0,N}(i,j)\,\bar{\psi}(i)\,\psi(j) \tag{3}$$

$$d_{0,N}(i,j) = d_0(i,j) - c(N), \qquad d_0(i,j) = \frac{1-n^{\alpha-1}}{1-n^{-\alpha}} d^{-\alpha}(i,j), \qquad i \neq j$$

$$d_0(i,i) = \frac{1-n^{-1}}{1-n^{-\alpha}}, \qquad c(N) = \frac{(1-n^{\alpha-1})^2}{(1-n^{-\alpha})(1-n^{-1})} n^{-\alpha(N+1)}$$

$$Z_{0,N} = \int \exp\{-H_{0,N}(\bar{\psi}, \psi, \alpha)\}\, d\psi\, d\bar{\psi}$$

Proof. We must prove that the matrix $(b(i,j))_{i,j \in \Lambda_N}$ is the inverse of $(d_{0,N}(i,j))_{i,j \in \Lambda_N}$. This means we must prove that

$$\sum_{j \in \Lambda_N} b(i,j)\, d_0(j,k) = \delta_{i,k} + c(N) \sum_{j \in \Lambda_N} b(i,j), \qquad i, k \in \Lambda_N \tag{4}$$

Let, for instance, $i \neq k$. It is easy to see that

$$\sum_{j \in \Lambda_N} b(i, j) = \frac{1 - n^{-1}}{1 - n^{\alpha - 2}} n^{N(\alpha - 1)}$$

To calculate the left part of (4) it is convenient to use the partition $\Lambda_N = A_1 \cup A_2 \cup A_3 \cup A_4$, where $A_1 = \{j \in \Lambda_N : s(i, j) > s(i, k)\}$, $A_2 = \{j \in \Lambda_N : s(i, j) = s(j, k) = s(i, k)\}$, $A_3 = \{j \in \Lambda_N : s(i, j) < s(i, k)\}$, and $A_4 = \{j \in \Lambda_N : s(j, k) < s(i, k)\}$. Let

$$I_m = \sum_{j \in A_m} b(i, j)\, d_0(j, k), \qquad m = 1, 2, 3, 4$$

Let us consider, for instance, I_1:

$$I_1 = \sum_{s = s(i,k) + 1}^{N} \sum_{j : s(i,j) = s} \frac{1 - n^{1 - \alpha}}{1 - n^{\alpha - 2}} n^{(\alpha - 2)s} \frac{1 - n^{\alpha - 1}}{1 - n^{-\alpha}} n^{-\alpha s}$$
$$= \frac{1 - n^{1 - \alpha}}{1 - n^{\alpha - 2}} \frac{1 - n^{\alpha - 1}}{1 - n^{-\alpha}} (n^{-(s(i,k) + 1)} - n^{-(N + 1)})$$

In similar way we obtain

$$I_2 = \frac{1 - n^{1 - \alpha}}{1 - n^{\alpha - 2}} \frac{1 - n^{\alpha - 1}}{1 - n^{-\alpha}} n^{-s(i,k)}(1 - 2n^{-1})$$

$$I_3 = -\frac{1 - n^{-1}}{1 - n^{\alpha - 2}} \frac{1 - n^{1 - \alpha}}{1 - n^{-\alpha}} n^{-s(i,k)}$$

$$I_4 = -\frac{1 - n^{-1}}{1 - n^{-\alpha}} \frac{1 - n^{\alpha - 1}}{1 - n^{\alpha - 2}} n^{-s(i,k)}$$

From all this there follows the validity of (4) for $i \neq k$.

Note that $Z_{0,N}^{-1} \exp\{-H_{0,N}(\bar{\psi}, \psi; \alpha)\}$ can be considered as the Grassmann-valued analog of a "density" function. The Hamiltonian (3) depends on N because it describes the restriction of the "Gaussian" state $\langle \cdot \rangle_0$ on the subvolume Λ_N.

The most general local potential which we can construct for the 4-component fermionic field is given by the form

$$U(\bar{\psi}, \psi; r, g) = r(\bar{\psi}_1 \psi_1 + \bar{\psi}_2 \psi_2) + g \bar{\psi}_1 \psi_1 \bar{\psi}_2 \psi_2$$

We define the Gibbs state ("expectation value") $\rho_N(r, g)$ on the $\mathfrak{A}_N$ as

$$(\rho_N(r, g))(F) = Z_N^{-1}(r, g) \langle F \exp\{-H_N\} \rangle_0$$
$$H_N(\bar{\psi}, \psi; r, g) = \sum_{i \in \Lambda_N} U(\bar{\psi}(i), \psi(i); r, g) \tag{5}$$
$$Z_N(r, g) = \langle \exp\{-H_N\} \rangle_0$$

$F \in \mathfrak{A}_N$ [We assume that $Z_N(r, g) \neq 0$]. If ρ is a state on $\mathfrak{A}_N$, then the renormalized state ρ' is defined on $\mathfrak{A}_{N-1}$ by

$$\rho'(F) = \rho(F(r_\alpha(\bar{\psi}, \psi)))$$

Theorem 1. Let $(r+1)^2 - g/n \neq 0$, $(r+1)^2 - g \neq 0$. Then

$$\rho'_N(r, g) = \rho_{N-1}(r', g')$$

where

$$r' = n^{\alpha-1}\left(\frac{(r+1)^2 - g}{(r+1)^2 - g/n}(r+1) - 1\right) \tag{6}$$

$$g' = n^{2\alpha-3}\left(\frac{(r+1)^2 - g}{(r+1)^2 - g/n}\right)^2 g \tag{7}$$

Proof. Let $(\bar{\psi}', \psi') = r_\alpha(\bar{\psi}, \psi)$ be given by (1). Let us consider the integral

$$\begin{aligned} I &= \langle F(\bar{\psi}', \psi') \exp\{-H_N(\bar{\psi}, \psi; r, g)\}\rangle_0 \\ &= Z_N(0,0)^{-1} \int F(\bar{\psi}', \psi') \\ &\qquad \times \exp\{-H_{0,N}(\bar{\psi}, \psi, \alpha) - H_N(\bar{\psi}, \psi; r, g)\}\, d\psi\, d\bar{\psi} \end{aligned}$$

where

$$d\psi\, d\bar{\psi} = \prod_{i \in \Lambda_N} d\psi(i)\, d\bar{\psi}(i) = \prod_{i \in \Lambda_N} d\psi_1(i)\, d\bar{\psi}_1(i)\, d\psi_2(i)\, d\bar{\psi}_2(i)$$

We introduce new variables $(\bar{\eta}(i), \eta(i))$, $i \in \Lambda_N$, by

$$\begin{aligned} (\bar{\psi}(i), \psi(i)) = n^{\alpha/2-1}&(\bar{\psi}'([(i-1)/n]+1), \psi'([(i-1)/n]+1)) \\ &+ (\bar{\eta}(i), \eta(i)) \end{aligned} \tag{8}$$

where $[\cdot]$ denotes an integer part. Using

$$\sum_{i \in V_{k,1}} (\eta(i), \bar{\eta}(i)) = 0, \qquad k \in \Lambda_{N-1}$$

we find that

$$H_{0,N}(\bar{\psi}, \psi, \alpha) = H_{0,N-1}(\bar{\psi}', \psi', \alpha) + \sum_{k \in \Lambda_{N-1}} \sum_{i \in V_{k,1}} \bar{\eta}(i)\, \eta(i) \tag{9}$$

So

$$I=(Z_N(0,0)\,c_0(n,N,\alpha))^{-1}$$
$$\times\int F(\bar{\psi}',\psi')\exp\{-H_{0,N-1}(\bar{\psi}',\psi',\alpha)\}$$
$$\times\prod_{k\in\Lambda_{N-1}}\left(\int\delta\left(\sum_{i\in V_{k,1}}(\bar{\eta}(i),\eta(i))\right)\exp\left\{-\left(\sum_{i\in V_{k,1}}\bar{\eta}(i)\,\eta(i)\right.\right.\right.$$
$$\left.\left.+\sum_{i\in V_{k,1}}U(n^{\alpha/2-1}(\bar{\psi}'(k),\psi'(k))+(\bar{\eta}(i),\eta(i));r,g)\right)\right\}$$
$$\left.\times\prod_{i\in V_{k,1}}d\eta(i)\,d\bar{\eta}(i)\right)\prod_{k\in\Lambda_{N-1}}d\psi'(k)\,d\bar{\psi}'(k)$$

$c_0(n,N,\alpha)$ is a "Berezinian" of the linear change of variables (8). Here $\delta(\bar{\eta},\eta)$ is a delta function defined by the condition

$$\int\delta(\bar{\eta},\eta)\,f(\bar{\eta},\eta)\,d\eta\,d\bar{\eta}=f(0,0)$$

The following simple integral representation for the δ-function will be convenient:

$$\delta(\bar{\eta},\eta)=\bar{\eta}_1\eta_1\bar{\eta}_2\eta_2=\int\exp(-(\bar{\xi}_1\eta_1+\cdots+\xi_2\bar{\eta}_2))\,d\xi\,d\bar{\xi}$$

Thus, we have

$$I=(Z_N(0,0)\,c_0(n,N,\alpha))^{-1}$$
$$\times\int F(\bar{\psi}',\psi')\exp\{-H_{0,N-1}(\bar{\psi}',\psi',\alpha)\}$$
$$\times\prod_{k\in\Lambda_{N-1}}\left(\int T((\bar{\psi}'(k),\psi'(k));\bar{\xi},\xi)^n\,d\xi\,d\bar{\xi}\right)$$
$$\times\prod_{k\in\Lambda_{N-1}}d\psi'(k)\,d\bar{\psi}'(k)$$

where

$$T(\bar{\psi}',\psi';\bar{\xi},\xi)=\int\exp\{-(\bar{\eta}\eta+\bar{\xi}\eta+\xi\bar{\eta}$$
$$+U(n^{\alpha/2-1}(\bar{\psi}',\psi')+(\bar{\eta},\eta);r,g))\}\,d\eta\,d\bar{\eta}$$

$\eta = (\eta_1, \eta_2)$, $\bar{\eta} = (\bar{\eta}_1, \bar{\eta}_2)$. By direct calculation one can verify the following relation:

$$\int \exp\{-(\bar{\xi}\eta + \xi\bar{\eta} + U((\bar{\eta}, \eta); r, g))\}\, d\eta\, d\bar{\eta}$$

$$= (r^2 - g) \exp\left\{-U\left(\bar{\xi}, \xi; \frac{r}{r^2 - g}, \frac{g}{(r^2 - g)^2}\right)\right\} \tag{10}$$

$r^2 - g \neq 0$. Introducing the variables

$$(\bar{\eta}', \eta') = n^{\alpha/2 - 1}(\bar{\psi}', \psi') + (\bar{\eta}, \eta)$$

$$\bar{\xi}' = \bar{\xi} - n^{\alpha/2 - 1}\bar{\psi}', \qquad \xi' = \xi + n^{\alpha/2 - 1}\psi'$$

and twice using (10), we obtain

$$\int T((\bar{\psi}', \psi'); \bar{\xi}, \xi)^n\, d\xi\, d\bar{\xi} = C(r, g) \exp\{-U((\bar{\psi}', \psi'); r', g')\}$$

where

$$C(r, g) = n((r+1)^2 - g)^{n-2}\,(n(r+1)^2 - g) \tag{11}$$

r' and g' are given by (6), (7). This proves the theorem.

It is easy to see that fixed points of the map $(r, g) \to (r', g')$, distinct from $r(\alpha) = g(\alpha) \equiv 0$, are described by the formulas

$$r_+(\alpha) = \frac{\sqrt{n} - n^{\alpha - 1}}{1 - \sqrt{n}}, \qquad g_+(\alpha) = \frac{r_+(\alpha)(1 + r_+(\alpha))^2}{1 + r_+(\alpha) + 1/\sqrt{n}}, \quad \alpha \neq 1,\ \alpha \neq 1/2 \tag{12}$$

$$r_-(\alpha) = \frac{-\sqrt{n} - n^{\alpha - 1}}{1 + \sqrt{n}}, \quad g_-(\alpha) = \frac{r_-(\alpha)(1 + r_-(\alpha))^2}{1 + r_-(\alpha) - 1/\sqrt{n}}, \quad \alpha \neq 1 \tag{13}$$

For $\alpha = 1$ all points of the tupe $g = 0$, $r \neq -1$ are also fixed points.

A simple calculation shows that the differential of the RG transformation on the "$\pm$" branches of fixed points is given by the matrix

$$D(r_\pm(\alpha), g_\pm(\alpha)) = \begin{pmatrix} \dfrac{2(1 \mp x)(1 \mp nx)\sqrt{n}}{(n-1)\,x} \pm \sqrt{n} & \pm \dfrac{4(1 \mp nx)^2\,(r_\pm + 1)}{(n-1)\,x} \\ -\dfrac{(1 \mp x)^2\,(r_\pm + 1)^{-1}\sqrt{n}}{(n-1)\,x} & \mp \dfrac{2(1 \mp x)(1 \mp nx)}{(n-1)\,x} + 1 \end{pmatrix} \tag{14}$$

where $x = n^{1/2 - \alpha}$.

The stability analysis is simplified by the observation that $\det(D(r_\pm(\alpha), g_\pm(\alpha)) = \pm\sqrt{n}$. From this analysis it follows that all points of the "+" branch with the exception of $\alpha = 3/2$ are hyperbolic; $\alpha = 3/2$ is a bifurcation value. At the point $(r_+(3/2), g_+(3/2)) = (0, 0)$ the "+" branch bifurcates from the trivial "Gaussian" one $[r(\alpha) = g(\alpha) \equiv 0]$. In addition, the spectrum of $D(r_+(\alpha), g_+(\alpha))$, $1/2 < \alpha < 3/2$, lies beyond the unit circle. All points of the "−" branch are hyperbolic also.

In the next section we shall find the range of the value α for which the thermodynamic limit of the models defined by the Hamiltonians $H_N(\bar{\psi}, \psi; r_\pm(\alpha), g_\pm(\alpha))$ exist. We note that the correlation functions $(\rho_N(r_\pm, g_\pm))(F)$, $F \in \mathfrak{A}_N$, are well defined only if $Z_N(r_\pm(\alpha), g_\pm(\alpha)) \neq 0$. Unlike the bosonic case, there exist a possibility of the vanishing of the statistical sum Z_N for real values of the coupling constants. From the proof of Theorem 1 it follows that

$$Z_N(r, g) = c_1(r, g; n, N, \alpha)\, Z_{N-1}(r', g') \tag{15}$$

where $c_1(r, g; n, N, \alpha) = c_2 C(r, g)^{n^{N-1}}$, c_2 does not depend on r and g. Therefore, if $Z_0(r, g) = 0$, then the same holds for $Z_N(r, g)$. Note that for "+" and "−" branches of the fixed points the factor $c_1(r, g; n, N, \alpha)$ is different from 0.

Direct calculation shows that

$$Z_0(r, g) = \left(\frac{1 - n^{\alpha-2}}{1 - n^{-1}}\right)^{-2} \left\{\left(r + \frac{1 - n^{\alpha-2}}{1 - n^{-1}}\right)^2 - g\right\}$$

The equation $Z_0(r_-(\alpha), g_-(\alpha)) = 0$ has no real root for $n \leqslant 4$. In all other cases the equations

$$Z_0(r_+(\alpha), g_+(\alpha)) = 0, \qquad Z_0(r_-(\alpha), g_-(\alpha)) = 0$$

have real roots

$$\alpha_+(n) = 2 - \log_n \frac{1 + 2\sqrt{n}}{2 + \sqrt{n}}, \qquad \alpha_-(n) = 2 - \log_n \frac{1 - 2\sqrt{n}}{2 - \sqrt{n}}$$

Therefore we shall exclude $\alpha = \alpha_+(n)$ from the description of the "+" branch and $\alpha = \alpha_-(n)$, $n > 4$, from the description of the "−" branch.

3. THE THERMODYNAMIC LIMIT

The thermodynamic limit of the fermionic model exists if all correlation functions (5) have a limit when $N \to \infty$. This means that we investigate the thermodynamic limit for the free boundary condition. One can

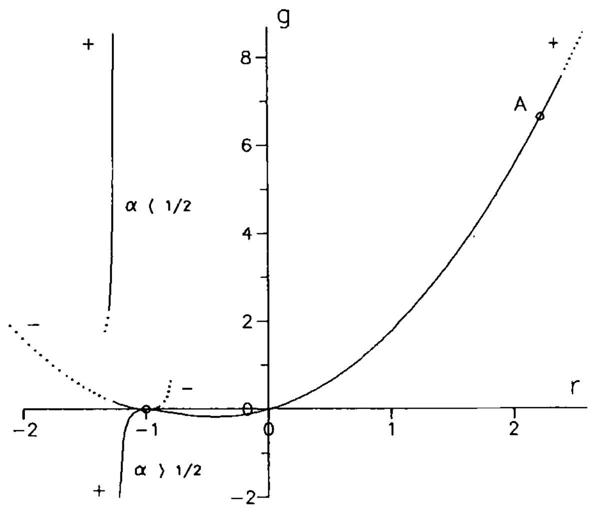

Fig. 1. The "+" and "−" branches of the RG fixed points ($n = 16$). The full lines correspond to the fixed points for which the thermodynamic limit exists. The point A is excluded as a zero of the statistical sum. The point $(-1, 0)$, corresponding to $\alpha = 1$, is excluded also because it lies on the critical parabola.

show[4,6] that in our model it will suffice to investigate the infinite-volume limit of the correlation functions

$$u_N^1(r, g) = \rho_N(r, g)(\psi_1(i)\,\bar\psi_1(i) + \psi_2(i)\,\bar\psi_2(i))$$

and

$$u_N^2(r, g) = \rho_N(r, g)(-\psi_1(i)\,\bar\psi_1(i)\,\psi_2(i)\,\bar\psi_2(i))$$

$i \in \Lambda_N$ (u^1 and u^2 do not depend on i). In the following it will be convenient to deal with the vector $u_N(r, g)$:

$$u_N(r, g) = \begin{pmatrix} u_N^1(r, g) \\ u_N^2(r, g) \end{pmatrix}$$

Lemma 2. We have

$$u_N(r, g) = A(r, g)\,u_{N-1}(r', g') + s(r, g) \tag{16}$$

where

$$A(r, g) = \frac{1}{n}\begin{pmatrix} \dfrac{\partial r'}{\partial r} & \dfrac{\partial g'}{\partial r} \\ \dfrac{\partial r'}{\partial g} & \dfrac{\partial g'}{\partial g} \end{pmatrix}, \qquad s(r, g) = \frac{1}{n}\begin{pmatrix} \dfrac{\partial \ln C(r, g)}{\partial r} \\ \dfrac{\partial \ln C(r, g)}{\partial g} \end{pmatrix}$$

Proof. Differentiating (15) with respect to r and dividing by the same equality (15), we find

$$\rho_N(r, g)\left(\sum_{i \in \Lambda_N} \psi_1(i)\,\bar{\psi}_1(i) + \psi_2(i)\,\bar{\psi}_2(i)\right)$$
$$= \frac{(\partial/\partial r)\{C^{n^{N-1}}(r, g)\}}{C^{n^{N-1}}(r, g)} \frac{Z_{N-1}(r', g')}{Z_{N-1}(r', g')}$$
$$+\rho_{N-1}(r', g')\left(\frac{\partial r'}{\partial r} \sum_{i \in \Lambda_{N-1}} \psi_1(i)\,\bar{\psi}_1(i) + \psi_2(i)\,\bar{\psi}_2(i)\right)$$
$$+\rho_{N-1}(r', g')\left(\frac{\partial g'}{\partial r} \sum_{i \in \Lambda_{N-1}} \psi_1(i)\,\bar{\psi}_1(i)\,\psi_2(i)\,\bar{\psi}_2(i)\right)$$

or

$$n^N u_N^1(r, g) = n^{N-1}\frac{\partial \ln C(r, g)}{\partial r}$$
$$+ n^{N-1}\left(\frac{\partial r'}{\partial r} u_{N-1}^1(r', g') + \frac{\partial g'}{\partial r} u_{N-1}^2(r', g')\right)$$

In a similar fashion we obtain the second row of (16).

Using (14), we see that

$$A(r_\pm, g_\pm) \equiv A_\pm = \begin{pmatrix} \dfrac{2(1 \mp x)(1 \mp nx)}{\sqrt{n}(n-1)\,x} \pm \dfrac{1}{\sqrt{n}} & \pm\dfrac{4(1 \mp nx)^2\,(r_\pm + 1)}{n(n-1)\,x} \\ -\dfrac{(1 \mp x)^2\,(r_\pm + 1)^{-1}}{\sqrt{n}(n-1)\,x} & \mp\dfrac{2(1 \mp x)(1 \mp nx)}{n(n-1)\,x} + \dfrac{1}{n} \end{pmatrix}$$

with $x = n^{1/2-\alpha}$.

Let $\lambda^\pm_{\max}(\alpha) = \max(|\lambda_1^\pm|, |\lambda_2^\pm|)$, where $\lambda_1^\pm$, $\lambda_2^\pm$ are the eigenvalues of the matrix $A_\pm$. We denote by $I_\pm(n)$ the range of α such that $\lambda^\pm_{\max}(\alpha) < 1$.

Lemma 3. For the "$-$" case $I_-(n) = 0$ if $n \leqslant 13$. In all other cases

$$I_\pm(n) = (1 - \Delta_n^\pm, 1 + \Delta_n^\pm)$$

where

$$\Delta_n^\pm = \log_n\{a_\pm + (a_\pm^2 - 1)^{1/2}\}, \qquad a_\pm = \frac{(n-1)^2}{4n} \pm \frac{n+1}{2\sqrt{n}}$$

Proof. We note that $\det(A_\pm) = \pm n^{-3/2}$. Now

$$\lambda_1^\pm = b_\pm + (b_\pm^2 \mp n^{-3/2})^{1/2}, \qquad \lambda_2^\pm = b_\pm - (b_\pm^2 \mp n^{-3/2})^{1/2}$$

where

$$b_{\pm}=\frac{1}{2}\left\{y_{\pm}\left(1\mp\frac{1}{\sqrt{n}}\right)\pm\frac{1}{\sqrt{n}}+\frac{1}{n}\right\},\quad y_{\pm}=\frac{2(1\mp x)(1\mp nx)}{\sqrt{n}(n-1)\,x}$$

If the eigenvalues of the matrix A_+ are nonreal, then they are conjugate and $\lambda^+_{\max}=n^{-3/4}<1$. Further we suppose that $b_+^2-n^{-3/2}\geqslant 0$.

Using the estimate

$$y_{\pm}\geqslant 2\,\frac{2\sqrt{n}\mp(n+1)}{\sqrt{n}(n-1)} \tag{17}$$

we obtain that $b_->0$ and $\min(\lambda_1^+,\lambda_2^+)=\lambda_2^+>2b_+>-1$. Therefore the inequality $\lambda^{\pm}_{\max}(\alpha)\geqslant 1$ is equivalent to

$$\lambda_1^{\pm}=b_{\pm}+\{b_{\pm}^2-\det(A_{\pm})\}^{1/2}\geqslant 1 \tag{18}$$

Thus, to find the range $\bar{I}_{\pm}(n)=\mathbb{R}\backslash I_{\pm}(n)$ it is necessary to solve the inequality (18) with the additional condition

$$b_{\pm}^2-\det(A_{\pm})\geqslant 0 \tag{19}$$

The system of inequalities (18), (19) is equivalent to the inequality

$$b_{\pm}\geqslant\{1+\det(A_{\pm})\}/2$$

After elementary transformations, it is be found that

$$y_{\pm}\geqslant 1-1/n \tag{20}$$

In the "$-$" case the inequality (20) follows from the inequality (17) for $n\leqslant 13$. In other cases the quadratic (in x) equation $y_{\pm}=1-1/n$ has roots $x_1=(1/\sqrt{n})\{a_{\pm}+(a_{\pm}^2-1)^{1/2}\}$, $x_2=1/nx_1$. The lemma is proved.

Let us denote $I'_+(n)=I_+(n)\backslash\{1,1/2,\alpha_+(n)\}$ and $I'_-(n)=I_-(n)\backslash\{1\}$. It is easy to check that for all $n>1$, $\alpha_+(n)\in I_+(n)$ and $\alpha_-(n)\notin I_-(n)$.

Theorem 2. The hierarchical fermion model, defined by the Hamiltonian $H_{0,N}(\bar{\psi},\psi,\alpha)+H_N(\bar{\psi},\psi;r_{\pm}(\alpha),g_{\pm}(\alpha))$, has a thermodynamic limit if and only if $\alpha\in I'_{\pm}(n)$.

Proof. One can show that

$$u_0(r,g)=\begin{pmatrix}\partial\ln Z_0/\partial r\\ \partial\ln Z_0/\partial g\end{pmatrix}$$

Iterating (16), we obtain

$$u_N(r_\pm, g_\pm) = A_\pm^N u_0^\pm + \sum_{i=0}^{N-1} A_\pm^i s_\pm \tag{21}$$

where

$$u_0^\pm \equiv u_0(r_\pm, g_\pm) = \frac{1 \pm 1/\sqrt{n}}{(r_\pm + 1)\{(2\sqrt{n} \pm n)\, x - 1/n \mp 2/\sqrt{n}\}} \times \begin{pmatrix} 2 \\ -(1 \pm 1/\sqrt{n})/(r_\pm + 1) \end{pmatrix}$$

$$s_\pm \equiv s(r_\pm, g_\pm) = \frac{1 \mp x}{(r_\pm + 1)\, n(n-1)\, x} \begin{pmatrix} 2\{nx \pm (n-2)\} \\ -\{x \pm (n-2)\}/(r_\pm + 1) \end{pmatrix}$$

The first part of the theorem follows from Eq. (21) and Lemma 3.

We can rewrite (21) as

$$u_N(r_\pm, g_\pm) = A_\pm^N \{u_0^\pm + (A_\pm - E)^{-1} s_\pm\} - (A_\pm - E)^{-1} s_\pm$$

If $A_\pm - E$ is degenerate, one can show that the model has no infinite-volume limit. Thus, if $\lambda_{\max}^\pm > 1$, then the thermodynamic limit exists only if

$$e = u_0^\pm + (A_\pm - E)^{-1} s_\pm$$

is the eigenvector of $A_\pm$ with eigenvalue less than 1. It is easy to see that the condition of the collinearity of the vector e and $A_\pm e$ leads to a bulky algebraic equation on x. For the "$-$" branch this equation has no positive real roots. For the "$+$" branch $x = n^{-1}$ ($\alpha = 3/2$) is the only admissible root, but $3/2 \in I'_+(n)$ for all n. The theorem is proved.

We see that the "$+$" branch bifurcates at the point $\alpha_0 = 3/2$ from the "Gaussian" trivial branch ($r \equiv 0, g \equiv 0$). It is interesting to note also that this branch can be described in terms of Wilson's $\varepsilon = 4 - d$ expansion (d is the dimensionality). Indeed, let us put $n = m^d$ and call d the dimension of the hierarchical lattice. We put $\alpha(d) = (d+2)/d$ (this choice corresponds to the Laplace kinetic term in the real case). Then

$$r_+(\alpha(d)) = \frac{m^{d/2} - m^2}{1 - m^{d/2}}, \qquad g_+(\alpha(d)) = \frac{r_+(\alpha(d))\{1 + r_+(\alpha(d))\}^2}{1 + r_+(\alpha(d)) + m^{-d/2}}$$

We see that this branch bifurcates from the "Gaussian" branch (d changes continuously) at $d = 4$. For $m \geqslant 3$, $\alpha(3) = 5/3 \in I'_+(n)$, but $\alpha(2) = 2 \notin I'(n)$ for any m.

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